Many to one matching: Externalities and stability

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Abstract

We analyze many to one matching problems in presence of external effects. We extend the notion of φ-stability (Sasaki and Toda (1996), "Two sided matching problems with externalities", Journal of Economic Theory, 70, 93-108) based on the concept of *estimation functions* $\boldsymbol{\varphi}$. We show that even under *full admissibility,* i.e. a situation where each matching is admissible for all agents, the set of φ-stable matchings may be empty. Hence, we provide a condition on agents' preferences, called *bottom substitutability,* that guarantees the existence of at least one φ-stable matching under *full admissibility.* Under the assumption that agents are *pessimistic enough*, we provide a set of *pessimistic estimation functions,* which depends on agents' preferences and implies that the assumption of *full admissibility* is neither neccesary nor sufficient for the existence of φ-stable matchings as is claimed by Sasaki and Toda (1996). We show that the set of φ-stable matchings and a notion of the core with non-myopic agents (φ-core) always coincide given any set of *estimation functions* $\boldsymbol{\varphi}$. Further, if preferences are *bottom substitutable* the φ-core always exists given the set of *pessimistic estimation functions*.

JEL classification numbers*: C78,* D62, D81. **Keywords***:* Many to one matching, externalities, non-myopic behavior, stability. **Research field***:* Game theory.

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1 Introduction.

In the standard two-sided matching model is commonly assumed that agents have complete and transitive preferences over the set of agents on the other side of the market, usually these preferences extent to preferences over matchings under the assumption that agents' valuations are independent each other, i.e. there is no group external effects. This assumption significantly reduces the complexity of the problem and make possible to derive many interesting results about the existence and properties of stable matchings. However, it is not difficult to find many non-trivial instances where seems unreasonable the absence of external effects.

In job market problems, firms' profits may depend on the group of workers hired by rival. For instance, pharmaceutical, automotive and in general technological firms hire research teams to develop and patent new products, the probability of introducing an innovation by a firm before its rivals depends on the quality of its researcher and researchers hired by rivals, i.e. expected profits of each firm may depend on the complete matching.

In the matching between universities and researchers, it seems reasonable that the quality of a university depends on the complete matching and not only the quality of its own researchers. Obviously, Harvard is the best university because it has the best group of researcher, however that does not mean that there is no other group of different researchers with equal or higher quality. This simply means that this group of researchers with higher quality than Harvard's ones is distributed among many other universities, instead of being matched with only one institution.

In professional sport leagues seems reasonable the presence of externalities among groups. Clubs in a professional league compete to win the tournament, the probability of winning the league may depends on the complete matching. For instance, if the FC Barcelona is matched with Xavi Hernandez, Andres Iniesta and Leo Messi any other club of the Spanish football league has a very low probability of winning the tournament, however if these three players would be matched with different teams, surely others clubs would improve their chance of winning the league.

The presence of external effects is problematic for at least two reasons. First of all, in order to capture external affects among groups, agents' preferences have to be specified over the set of all possible matchings instead of agents on the other side of the market. Hence, preferences over matchings are not equivalent to preferences over firms (or subsets ow workers). For instance, for each worker there is no a unique way to order firms and the possibility of being unmatched consistent with preferences over matchings, whether a partner is acceptable or not depends on how are matched the others. Then constrains over preferences that guarantee the existence of stable matchings in the standard setup are not well defined and cannot be applied in a matching problem with externalities.

Another problem regards the definition of equilibrium in presence of externalities. In the standard setup agents deviate in a myopic way, they do not take into account how the others react against a deviation. In presence of externalities the assumption of myopic behavior is not reasonable, since it is possible that deviating will not be finally profitable. Each agent has to predict how the others react against deviations, therefore different behavioral assumptions lead to different notions of the equilibrium.

Sasaki and Toda (1996) were the first in analyzing the problem of external effects in the marriage problem. They use a notion of equilibrium similar to the idea of the conjectural equilibrium. An outcome is a conjectural equilibrium if no agent has an incentive to deviate under a given conjecture about the reactive behaviors of the others (Rubinstein and Wolinsky, 1994; Azrieli, 2009). In this sense, agents predict which matching are admissible outcomes based on a certain conjecture about agents' behavior, Sasaki and Toda (1996) call to this predictions *estimation functions.* Given a set of *estimation functions* $\boldsymbol{\varphi}$, a matching is φ-stable if it is admissible for all agents and it is not blocked by any pair or individual agent. They claim that *estimation function* in which any matching is admissible for all agents, say *full admissibility*, is necessary and sufficient for the general existence of φ-stable matchings.

The assumption of *full admissibility* implies that even irrational matchings are taken into account by agents when they make their estimations. Hafalir (2008) provides a method to find a set of *estimation functions* that depends on agents' preferences. This set of *estimation functions* guarantees the existence of φ-stable matchings but depends crucially on the fact that any standard marriage model has a nonempty set of stable matchings (Gale and Shapley, 1962).

This paper deals about many to one matching problems with externalities. We show that *full admissibility* is neither necessary nor sufficient to ensure the existence of φ-stable matchings in this setup. Further, if we relax the setup by assuming that there are externalities only on the firms' side, it is possible to find problems with no φ-stable matchings. Given this problem, we deduce a condition of weak substitutability of preferences called *bottom substitutability* that guarantees the non-emptiness of the set of φ-stable matchings under *full admissibility*.

After that, we analyze the problem without the assumption of *full admissibility*. We argue that is not possible to apply Hafalir's (2008) model, since some instances of the standard many to one problem may have no stable matchings. Under the assumption that agents are *pessimistic* enough, we determine a set of *pessimistic estimation functions* which depend on agents' preferences. Given the set of *pessimistic estimation functions* there exists at least one φ-stable matching whenever agents' preferences satisfy *bottom substitutability*. Further, we show that this set of *pessimistic estimation functions* implies that *full admissibility* is not a necessary condition for the general existence of φ-stable matchings. After that, we show that all of these results extend to the marriage problem.

Sasaki and Toda (1996) and Hafalir (1996) argue that, in general, the core and the set of pair-wise φ-stable matching do not coincide in presence of externalities. Further, Sasaki and Toda (1996) claim that the core may be empty. These authors analyze a notion of the core which implies that agents may behave in a myopic way when they deviate with large coalitions. However, in presence of externalities, it seems more reasonable that agents behave in a non-myopic way independently of the size of deviating coalitions.

We propose a notion of the core considering that agents always behave in a non-myopic way. We prove that given any set of *estimation functions* $\boldsymbol{\varphi}$, the set of φ-stable matchings and the core with non-myopic agents, say the φ-core, coincide. Further, given the set of *pessimistic estimation functions* the φ-core is always non-empty.

The rest of the paper is organized as follows: in section two, we describe the model; in section three, we define *bottom substitutability* and it is characterized the existence of φ-stable matchings under *full admissibility*; in section four, we introduce set of *pessimistic estimation functions* and we analyze φ-core; in section five, we extend results to the marriage problem; in section six, we present some conclusions. Finally, all proofs are in the appendix in section seven.

2 The model.

There are two disjoint and finite sets of agents, a set of $m \geq 1$ firms $\mathbf{F} = \{f_1, ...f_m\}$ and a set of $n \geq 2$ workers $\mathbf{W} = \{w_1, ...w_n\}$. Generic elements of both sets are denoted, respectively, by f and by w. A generic agent is denoted by $a \in \mathbf{F} \cup \mathbf{W}$, whereas a generic subset of workers is denoted by $S \subset \mathbf{W}$. Firms will hire sets of workers and workers will work for at most one firm. Each firm $f \in \mathbf{F}$, has a quota $q_f \leq n$ that denotes the maximum number of workers that f can hire. For each $f \in \mathbf{F}$, let $H_f = \{S \subset \mathbf{W} : |S| \leq q_f\}$ denote the set of all subsets of workers (including the empty set) that f is able to hire.

Each $f \in \mathbf{F}$ has a complete, strict and transitive preference order P_f over the set 2^W, such that $\phi P_f S$ for all $S \notin H_f$. Similarly, each $w \in \mathbf{W}$ has a complete, strict and transitive preference order P_w over the set $\mathbf{F} \cup \{w\}$. For each $a \in \mathbf{F} \cup \mathbf{W}$, R_a denotes the weak preference relation associated to P_a. Let $\mathbf{P} = (P_{f_1}, ..., P_{f_m}; P_{w_1}, ..., P_{w_n})$ denote the agents' preference profiles.

The problem consists of matching workers with firms, allowing for the possibility that some agents may remain unmatched. Formally,

Definition 1 *A matching μ is a mapping from the set $\mathbf{F} \cup \mathbf{W}$ into the set of all subsets of $\mathbf{F} \cup \mathbf{W}$ such that:*

1. $|\mu(w)| = 1$ *for all* $w \in \mathbf{W}$ *and either* $\mu(w) \cap \mathbf{F} \neq \phi$ *or* $\mu(w) = \{w\}$*;*
2. $\mu(f) \in H_f$ *for all* $f \in \mathbf{F}$*; and*
3. $\mu(w) = \{f\}$ *if and only if* $w \in \mu(f)$.

Let $\mathcal{M}$ be the set of all possible matchings given $\mathbf{F}$ and $\mathbf{W}$. A matching problem is a three-tuple $(\mathbf{F}, \mathbf{W}, \mathbf{P})$.

A matching μ is blocked by a worker $w \in \mathbf{W}$ if $w P_w \mu(w)$. A coalition firm-set of workers (f, S) such that $S \in H_f$ and $\mu(f) \neq S$, blocks the matching μ if: 1. $S P_f \mu(f)$ and 2. $f R_w \mu(w)$ for all $w \in S$. A matching is stable if is not blocked by any worker and any coalition (f, S). Let $\mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P})$ denote the set of stable matchings of the problem.

Let P_f be a preference relation of firm f, let $Ch_f : 2^W \to H_f$ denotes the choice function of firm f. For each $f \in \mathbf{F}$ and any $S \subset \mathbf{W}$, $Ch_f(S) \subset S$ and $Ch_f(S) R_f S'$ for all $S' \subset S$. Given a problem $(\mathbf{F}, \mathbf{W}, \mathbf{P})$ the preference profile $\mathbf{P}$ satisfy substitutability if, for each $f \in \mathbf{F}$ and any $S \subset \mathbf{W}$ such that $w, w' \in S$, if $w \in Ch_f(S)$ then $w \in Ch_f(S \setminus \{w'\})$. When preferences are substitutable the set of stable matchings is always nonempty (Theorem 6.5, Roth and Sotomayor, 1990).

Remark 1 *The set of pair-wise stable matchings is equal to the core defined by weakly domination (Proposition 6.4, Roth and Sotomayor, 1990). In addition, the core defined by weakly domination, is equal to the set of group-stable matchings with a definition of blocking group equivalent to ours (Echeniche and Oviedo, 2004). Hence,* $\mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P})$ *is equal to the set of pair-wise stable matchings and is not empty, when firms have substitutable preferences and all preferences are strict.*

For each $a \in \mathbf{F} \cup \mathbf{W}$, P_a^* denotes a complete, strict and transitive preference order over the set $\mathcal{M}$. For instance, given any two matchings $\mu, \mu' \in \mathcal{M}$, $\mu P_f^* \mu'$ means that the firm f prefers the matching μ to μ'. Note that no firm (worker) is indifferent between two different matchings that assign the same subset of workers (the same firm), this captures the presence of externalities in the problem. For each $a \in \mathbf{F} \cup \mathbf{W}$, R_a^* denotes the weak preference relation associated to P_a^*. Let $\mathbf{P}^* = \left(P_{f_1}^*, ..., P_{f_m}^*; P_{w_1}^*, ..., P_{w_n}^*\right)$ denote the agents' preference profile. A matching problem with externalities is a three-tuple $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

For each (f, S) such that $S \in H_f$, let $A(f, S) = \{\mu \in \mathcal{M} : \mu(f) = S\}$ denote the set of all matchings where f and S are matched. The set $A(w, a)$ is defined in a similar way for each (w, a) such that $a \in \mathbf{F} \cup \{w\}$.

Since there is no mechanism to match agents and the presence of externalities, agents estimate which matchings are admissible outcomes given a conjecture about the reactive behavior of the others. Formally, for each firm f and any $S \in H_f$ (each w and any $a \in \mathbf{F} \cup w$), let $\varphi_f(S) \subset A(f, S)$ ($\varphi_w(a) \subset A(w, a)$) denote the *estimation functions* of firm f (of w). $\varphi_f(S)$ is the non-empty subset of all admissible matchings where firm f (the worker w) is matched with workers S (the agent a). Intuitively, each firm f (worker w) believes that each matching in its estimation function $\varphi_f(S)$ ($\varphi_w(a)$) occurs with positive probability. Given a problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, let $\boldsymbol{\varphi} = \left\{\left(\varphi_f(\cdot), \varphi_w(\cdot)\right) : f \in \mathbf{F} \text{ and } w \in \mathbf{W}\right\}$ denote a set of *estimation functions*. Note that *estimation functions* may be exogenously given or may depend on agents preferences.

Definition 2 *Given a set of estimation functions* $\boldsymbol{\varphi}$*, a matching* μ *is* φ*-admissible in the problem* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$ *if* $\mu \in \varphi_{a}(\mu(a))$ *for all* $a \in \mathbf{F} \cup \mathbf{W}$*.*

A matching μ is φ-admissible if it is admissible for all agents, only φ-admissible matchings are candidates for stability. If some matching does not satisfy φ-admissibility, at least one agent considers that this matching will occur with probability zero and as consequence does not consider neither deviations from that matching nor if that is part of a profitable deviation. Since each agent believes that any matching in its *estimation functions* is likely, a coalition firm-set of workers blocks a matching only if all deviating agents are better off under all admissible matchings after deviating. Formally,

Definition 3 *An individual worker* $w \in \mathbf{W}$*, such that* $\mu(w) \neq w$*, blocks the matching* μ *if:* $\acute{\mu} P_{w}^{*} \mu$ *for all* $\acute{\mu} \in \varphi_{w}(w)$*.*

Definition 4 *A coalition firm-set of workers* (f, S) *such that* $S \in H_{f}$ *and* $\mu(f) \neq S$*, blocks the matching* μ *in the problem* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$ *if:*

1. $\mu' P_{f}^{*} \mu$ *for all* $\mu' \in \varphi_{f}(S)$ *and*
2. $\mu'' P_{w}^{*} \mu$ *for all* $\mu'' \in \varphi_{w}(f)$ *and all* $w \in S$*.*

Given a set of *estimation functions* $\boldsymbol{\varphi}$ a matching μ is φ-stable if, it is φ-admissible and is not blocked by any individual worker and any coalition (f, S). Let $\mathbb{E}_{\varphi}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$ denote the set of φ-stable matchings.

2.1 Motivating examples.

In Introduction, we argue that it is not difficult to find instances of the matching problem where the presence of group externalities is very reasonable. Such instances regard job matching problem, the matching between universities and researchers and matching markets where firms compete for winning a prize, for instance professional sport leagues, tournaments, etc. In this section, we abstract from a particular instance and we analyze some simple examples that are useful to set our problem. In what follows, we show that there exist instances of the many to one matching with no φ-stable matching under *full admissibility.* We say that the set *estimation functions* $\boldsymbol{\varphi}$ satisfies *full admissibility* if for each $a \in \mathbf{F} \cup \mathbf{W}$, $\varphi_{a}(\cdot) = A(a, \cdot)$, i.e. any matching is φ-admissible. Let $\mathbb{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$ denote the set of φ-stable matchings under *full admissibility.* Sasaki and

Toda (1996) analyze the one to one matching problem and claim that *full admissibility* is necessary and sufficient for the general existence of φ-stable matchings.

Example 1 *Consider a 3x3 marriage problem. We restrict agents preferences over the set of matchings with no unmatched agents. The set of matchings is given in the next table, we denote the partner of each man in the order* m_1, m_2, m_3.

Table 1

$\mu_1 = w_1, w_2, w_3$	$\mu_2 = w_1, w_3, w_2$
$\mu_3 = w_2, w_1, w_3$	$\mu_4 = w_2, w_3, w_1$
$\mu_5 = w_3, w_1, w_2$	$\mu_6 = w_3, w_2, w_1$

Preferences are given by the next orders,

$P^*_{m_1} = \mu_4, \mu_1, \mu_2, \mu_3, \mu_5, \mu_6.$

$P^*_{m_2} = \mu_6, \mu_4, \mu_3, \mu_1, \mu_5, \mu_2.$

$P^*_{m_3} = \mu_1, \mu_3, \mu_2, \mu_4, \mu_6, \mu_5.$

$P^*_{w_1} = \mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6.$

$P^*_{w_2} = \mu_3, \mu_4, \mu_1, \mu_2, \mu_5, \mu_6.$

$P^*_{w_3} = \mu_3, \mu_1, \mu_4, \mu_6, \mu_5, \mu_2.$

Suppose that any matching is admissible for each agent but m_1*, assume that* $\varphi_{m_1}(w_2) = \{\mu_4\} \neq \{\mu_3, \mu_4\} = A(m_1, w_2)$. *Note that* μ_3 *is the less preferred matching for* m_1 *in* $A(m_1, w_2)$. *It is not difficult to show that* μ_2, μ_4, μ_5 *and* μ_6 *are blocked by* (m_3, w_3)*;* μ_1 *is blocked by* (m_1, w_2) *and* μ_3 *is blocked by* (m_1, w_1). *Hence the set of pair-wise* φ*-stable matchings is empty.*

Assume that there are externalities only on the firms' side of the market, i.e. firms have preferences over the set of all possible matchings whereas workers have preferences over the set of firms and the possibility of remaining unmatched. This situation is reasonable if worker's preferences depend mostly on salaries and firms' preferences depend on the set of workers hired by other firms. For instance, in sport professional leagues the probability of a club of winning the tournament depends clearly on the complete matching. A club has more possibilities of winning the tournament if there is no other club that has at the same time the best defense and the best attack of the league. In the next example, we show that under *full admissibility* there could be no φ-stable matching in the many to one setup.

Example 2 *Consider a matching problem with two firms and three workers,* $\mathbf{W} = \{w_1, w_2, w_3\}$, $\mathbf{F} = \{f_1, f_2\}$ *and* $q_{f_1} = q_{f_2} = 2$. *The set of matching is described in the next table,*

Table 2

$\mu_1 = f_1, f_1, f_2$	$\mu_2 = f_1, f_2, f_2$	$\mu_3 = f_1, f_2, f_1$	$\mu_4 = f_1, f_1, w_3$
$\mu_5 = f_1, w_2, w_3$	$\mu_6 = f_1, w_2, f_1$	$\mu_7 = f_1, w_2, f_2$	$\mu_8 = f_1, f_2, w_3$
$\mu_9 = f_2, f_2, f_1$	$\mu_{10} = f_2, f_1, f_1$	$\mu_{11} = f_2, f_1, f_2$	$\mu_{12} = f_2, f_2, w_3$
$\mu_{13} = f_2, w_2, w_3$	$\mu_{14} = f_2, w_2, f_2$	$\mu_{15} = f_2, w_2, f_1$	$\mu_{16} = f_2, f_1, w_3$
$\mu_{17} = w_1, w_2, f_2$	$\mu_{18} = w_1, f_1, w_3$	$\mu_{19} = w_1, f_2, w_3$	$\mu_{20} = w_1, f_1, f_1$
$\mu_{21} = w_1, f_2, f_2$	$\mu_{22} = w_1, f_1, f_2$	$\mu_{23} = w_1, f_2, f_1$	$\mu_{24} = w_1, w_2, f_1$
$\mu_{25} = w_1, w_2, w_3$			

Suppose that the firms and workers have the next preferences:

$P^*_{f_1} = \mu_6, \mu_3, \mu_4, \mu_1, \mu_{20}, \mu_{10}, \mu_7, \mu_5, \mu_8, \mu_2, \mu_{22}, \mu_{16}, \mu_{18}, \mu_{11}, \mu_{25}, \mu_{19}, \mu_{17}, \mu_{14}, \mu_{13}, \mu_{12}, \mu_{21}, \mu_{23}, \mu_{24}, \mu_{15}, \mu_9.$

$P^*_{f_2} = \mu_{14}, \mu_{11}, \mu_{21}, \mu_2, \mu_{12}, \mu_9, \mu_7, \mu_{22}, \mu_{17}, \mu_1, \mu_{13}, \mu_{16}, \mu_{15}, \mu_{10}, \mu_{19}, \mu_8, \mu_{23}, \mu_3, \mu_{25}, \mu_5, \mu_{20}, \mu_6, \mu_{18}, \mu_{24}, \mu_4.$

$P_{w_1} = f_2, f_1, w_1.$

$P_{w_2} = f_2, f_1, w_2.$

$P_{w_3} = f_1, f_2, w_3.$

Assume that any matching is φ*-admissible in the problem. This implies that any matching is a candidate for* φ*-stability,*

$\mu_3 = f_1, f_2, f_1$ *is blocked by* (f_2, w_1);

$\mu_1 = f_1, f_1, f_2$ *is blocked by* $(f_2, \{w_1, w_3\})$;

$\mu_{10} = f_2, f_1, f_1$ *is blocked by* $(f_2, \{w_1, w_2\})$;

$\mu_{11} = f_2, f_1, f_2$ *is blocked by* $(f_1, \{w_2, w_3\})$;

$\mu_2 = f_1, f_2, f_2$ *is blocked by* $(f_2, \{w_1, w_3\})$ *and*

$\mu_9 = f_2, f_2, f_1$ *is blocked by* (f_1, ϕ).

On the other hand, any matching that leaves w_1 *unmatched is blocked either by* $(f_1, \{w_1\})$ *or by* $(f_2, \{w_1\})$. *Any matching that leaves* w_2 *unmatched is blocked either by* $(f_1, \{w_2\})$, $(f_2, \{w_2\})$ *or* $(f_2, \{w_2, w_3\})$. *Finally, any matching that leaves* w_3 *unmatched is blocked by* $(f_2, \{w_1, w_3\})$. *Then* $\mathbb{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*) = \phi$.

With externalities on both sides of the market, it is more difficult that a coalition deviates from a matching. For instance in the previous example the coalition

$(f_2, \{w_1, w_3\})$ blocks the matching μ_1, but worker w_3 is already matched with f_2 under μ_1. Hence $(f_2, \{w_1, w_3\})$ cannot block μ_1 since w_3 will not be better off under all admissible matching after deviating. However, it is possible to find instances where the set of φ-stable matchings is empty, as we show in the next example.

Example 3 *Consider a matching problem with two firms and three workers,* $\mathbf{W} = \{w_1, w_2, w_3\}$*,* $\mathbf{F} = \{f_1, f_2\}$*,* $q_{f_1} = 2$ *and* $q_{f_2} = 1$*. The set of matching is given by:*

Table 3

$\mu_1 = f_1, f_1, f_2$	$\mu_2 = f_1, w_2, f_2$	$\mu_3 = w_1, f_1, f_2$	$\mu_4 = w_1, w_2, f_2$
$\mu_5 = f_1, f_1, w_3$	$\mu_6 = f_1, w_2, w_3$	$\mu_7 = w_1, f_1, w_3$	$\mu_8 = f_1, f_2, f_1$
$\mu_9 = w_1, f_2, f_1$	$\mu_{10} = f_1, f_2, w_3$	$\mu_{11} = w_1, f_2, w_3$	$\mu_{12} = f_1, w_2, f_1$
$\mu_{13} = w_1, w_2, f_1$	$\mu_{14} = f_2, f_1, f_1$	$\mu_{15} = f_2, f_1, w_3$	$\mu_{16} = f_2, w_2, f_1$
$\mu_{17} = f_2, w_2, w_3$	$\mu_{18} = w_1, f_1, f_1$	$\mu_{19} = w_1, w_2, w_3$	

Consider the next preferences and assume that any matching is φ*-admissible.*

$P^*_{f_1} = \mu_1, \mu_5, \mu_4, \mu_{11}, \mu_{17}, \mu_{19}, \mu_{10}, \mu_2, \mu_6, \mu_7, \mu_3, \mu_{15}, \mu_{13}, \mu_{16},$
$\mu_9, \mu_{14}, \mu_{18}, \mu_8, \mu_{12}.$

$P^*_{f_2} = \mu_2, \mu_3, \mu_4, \mu_1, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{17}, \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_5, \mu_6,$
$\mu_7, \mu_{12}, \mu_{13}, \mu_{18}, \mu_{19}.$

$P^*_{w_1} = \mu_1, \mu_2, \mu_5, \mu_6, \mu_8, \mu_{10}, \mu_{12}, \mu_{17}, \mu_{16}, \mu_{15}, \mu_{14}, \mu_3, \mu_4, \mu_7,$
$\mu_9, \mu_{11}, \mu_{13}, \mu_{18}, \mu_{19}.$

$P^*_{w_2} = \mu_8, \mu_9, \mu_{10}, \mu_{11}, \mu_{18}, \mu_{15}, \mu_{14}, \mu_7, \mu_5, \mu_3, \mu_1, \mu_{19}, \mu_{17}, \mu_{16},$
$\mu_{13}, \mu_{12}, \mu_6, \mu_4, \mu_2.$

$P^*_{w_3} = \mu_8, \mu_9, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{16}, \mu_{18}, \mu_5, \mu_6, \mu_7, \mu_{10}, \mu_{11}, \mu_{15}, \mu_{19},$
$\mu_{17}, \mu_1, \mu_2, \mu_3, \mu_4.$

It is not difficult to check that:

a) w_3 *blocks the matchings:* $\mu_1, \mu_2, \mu_3, \mu_4$*;*

b) $(f_2, \{w_2\})$ *blocks the matchings:* $\mu_5, \mu_6, \mu_7, \mu_{12}$*;*

c) (f_1, ϕ) *blocks the matchings:* $\mu_8, \mu_9, \mu_{10}, \mu_{14}, \mu_{15}$*;*

d) $(f_2, \{w_1\})$ *blocks the matchings:* μ_{11}, μ_{18} *and*

e) $(f_1, \{w_1, w_2\})$ *blocks the matchings:* $\mu_{13}, \mu_{16}, \mu_{17}, \mu_{19}$*.*

Hence, the set of φ*-stable matchings is empty.*

This simple example show that *full admissibility* is not sufficient to guarantee the existence of φ-stable matchings. This contrasts with the result of Sasaki and Toda

(1996) in the one to one setup. In the next section, we provide a condition on agents' preferences called bottom substitutability that guarantees the existence of φ-stable matchings under *full admissibility.*

3 The existence of φ-stable matchings under *full admissibility.*

It is well known that substitutability guarantees the existence of stable matchings in the standard many to one setup. However, in general, this constrain over agents' preferences is not compatible with the presence of group externalities.

First of all, firms' choice functions might not be well defined because firms' preferences are defined over the set of matchings and this implies that there is no a unique way to order subsets of workers. In general, it is not possible to impose constrains over agents' preferences like substitutability, since these conditions require a well defined preference order over subsets of workers for each firm.

Further, a direct analysis of agents' preferences is a very difficult task, since agents' preferences are complete orders over the set of all possible matchings. Alternatively, given any matching problem with externalities, it is possible to derive a consistent *reduced problem* without externalities that allows us to apply many results from the standard setup. This approach is useful if a *reduced problem* is always well defined for any instance of the original problem and if the existence of equilibrium allocations in the *reduced problem* implies the existence of equilibrium allocations in the original problem (Shapley and Shubik, 1969; Sasaki and Toda,1996; Hafalir, 2008; Klaus and Klijn, 2005).

3.1 The *reduced problem.*

In this section, we introduce a method to construct a consistent *reduced problem* for any instance of the matching problem with externalities. For any coalition (f, S) such that $S \in H_f$, the matching $\mu_{f,S}$ satisfies: a) $\mu_{f,S} \in A(f, S)$ and b) $\acute{\mu} R^*_f \mu_{f,S}$ for all $\acute{\mu} \in A(f, S)$. Similarly for any worker-firm pair (w, f) the matching $\mu_{w,f}$ satisfies: a) $\mu_{w,f} \in A(w, f)$ and b) $\acute{\mu} R^*_w \mu_{w,f}$ for all $\acute{\mu} \in A(w, f)$. $\mu_{f,S}$ denotes the f's worse possible matching where f and $S \in H_f$ are matched. Note that this matching is uniquely defined, since preferences are strict and complete. Analogously, given a set of *estimation functions* $\boldsymbol{\varphi}$, the matching μ_f^S (μ_w^f) satisfies: a) $\mu_f^S \in \varphi_f(S)$ ($\mu_w^f \in \varphi_w(f)$) and b) $\mu' R_f^* \mu_f^S$ for all $\mu' \in \varphi_f(S)$ ($\mu'' R_w^* \mu_w^f$ for all $\mu'' \in \varphi_w(f)$). Note that the assumption of *full admissibility* implies $\mu_{f,S} = \mu_f^S$ for each firm and $\mu_{w,f} = \mu_w^f$ for each worker.

Given any instance of the problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ and a set of *estimation functions* $\boldsymbol{\varphi}$, for each $f \in \mathbf{F}$, the preference order P_f^{φ} over 2^W is defined as follows: for any $S, S' \in 2^W$, $SP_f^{\varphi}S'$ if and only if $\mu_f^S P_f^* \mu_f^{S'}$. Similarly for each $w \in \mathbf{W}$, given any $f, f' \in \mathbf{F}$, $fP_w^{\varphi}f'$ if and only if $\mu_w^f P_w^* \mu_w^{f'}$.

Since for each $a \in \mathbf{F} \cup \mathbf{W}$, the preference order P_a^* is complete and strict over $\mathcal{M}$, implies that P_a^{φ} is well defined, in the sense, that P_a^{φ} is a complete, strict and transitive preference order over, respectively, the sets $\mathbf{F} \cup \{w\}$ and 2^W. Hence, given any problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, this method induces a well defined matching problem without externalities denoted by $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$.

Example 4 *Consider the problem analyzed in Example 2. Take subsets of workers* $\{w_1, w_3\}$ *and* $\{w_1, w_2\}$*. It is easy to verify that* $\mu_{f_1}^{\{w_1,w_3\}} = \mu_3$ *and* $\mu_{f_1}^{\{w_1,w_2\}} = \mu_1$*, hence* $\mu_3 P_{f_1}^* \mu_1$ *implies* $\{w_1, w_3\} P_{f_1}^{\varphi} \{w_1, w_2\}$*. The complete preference profile is,*

$P_{f_1}^{\varphi} = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \phi, \{w_3\}.$

$P_{f_2}^{\varphi} = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \phi.$

We make the same exercise for Example 3, in this case we have,

$P_{f_1}^{\varphi} = \{w_1, w_2\}, \phi, \{w_1\}, \{w_2\}, \{w_3\}, \{w_2, w_3\}, \{w_1, w_3\}.$

$P_{f_2}^{\varphi} = \{w_3\}, \{w_1\}, \{w_2\}, \phi.$

$P_{w_1}^{\varphi} = f_1, f_2, w_1.$

$P_{w_2}^{\varphi} = f_2, f_1, w_2.$

$P_{w_3}^{\varphi} = f_1, w_3, f_2.$

The next result establishes that any stable matching in the *reduced problem* is φ-stable in the original one whereas those stable matchings are φ-admissible.

Proposition 1 *Let* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ *be any matching problem with externalities and* $\boldsymbol{\varphi}$ *any set of estimation functions. If a matching* μ *satisfies: a)* $\mu \in \mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$ *and b)* $\mu \in \varphi_a(\mu(a))$ *for all* $a \in \mathbf{F} \cup \mathbf{W}$*, then* $\mu \in \mathbb{E}_{\varphi}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$*.*

Under *full admissibility* any matching is φ-admissible, hence a direct implication of the previous result is that any stable matching in the *reduced problem* is φ-stable.

Corollary 1 *Let* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ *be any matching problem with externalities. Assume that the set of estimation functions* $\boldsymbol{\varphi}$*, satisfies full admissibility, i.e. 1.* $\varphi_f(S) = A(f, S)$ *for all* f *such that* $S \in H_f$ *and* $\varphi_w(a) = A(w, a)$ *for all* w *such that* $a \in F \cup w$*, then* $\mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi}) \subset \mathbb{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

The existence of stable matchings in the *reduced problem* implies the existence of φ-stable matchings under *full admissibility*, however the converse of Proposition 1 does not hold as we show in the next example.

Example 5 *Consider a matching problem with two firms and three workers,* $\mathbf{W} = \{w_1, w_2, w_3\}$*,* $\mathbf{F} = \{f_1, f_2\}$ *and* $q_{f_1} = q_{f_2} = 2$*. Firms' preferences are given in Example 2, whereas workers' preferences are the next,*

$P^*_{w_1} = \mu_9, \mu_{14}, \mu_{15}, \mu_{16}, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{10}, \mu_1, \mu_8, \mu_7, \mu_4, \mu_5, \mu_6, \mu_3, \mu_2, \mu_{17}, \mu_{23}, \mu_{24}, \mu_{25}, \mu_{22}, \mu_{18}, \mu_{21}, \mu_{19}, \mu_{20}.$

$P^*_{w_2} = \mu_{23}, \mu_{21}, \mu_{19}, \mu_{12}, \mu_9, \mu_8, \mu_3, \mu_2, \mu_{22}, \mu_{20}, \mu_{18}, \mu_{16}, \mu_{11}, \mu_4, \mu_1, \mu_{10}, \mu_{25}, \mu_{24}, \mu_{17}, \mu_{15}, \mu_{13}, \mu_7, \mu_6, \mu_5, \mu_{14}.$

$P^*_{w_3} = \mu_{24}, \mu_{23}, \mu_{20}, \mu_{15}, \mu_9, \mu_6, \mu_3, \mu_{10}, \mu_{22}, \mu_{21}, \mu_{17}, \mu_{14}, \mu_{11}, \mu_7, \mu_1, \mu_2, \mu_{25}, \mu_{19}, \mu_{18}, \mu_{16}, \mu_{13}, \mu_8, \mu_5, \mu_4, \mu_{12}.$

The reduced problem has the next preference profile,

$P^{\varphi}_{f_1} = \{w_1, w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_2\}, \phi, \{w_3\}.$

$P^{\varphi}_{f_2} = \{w_1, w_3\}, \{w_2, w_3\}, \{w_1, w_2\}, \{w_3\}, \{w_1\}, \{w_2\}, \phi.$

$P^{\varphi}_{w_1} = f_2, f_1, w_1.$

$P^{\varphi}_{w_2} = f_2, f_1, w_2.$

$P^{\varphi}_{w_3} = f_1, f_2, w_3.$

It is not difficult to show that there is no stable matching in the reduced problem, i.e. $\mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi}) = \phi$*, whereas* $\mu_{11} = f_2, f_1, f_2$ *is* φ*-stable under full admissibility.*

Note that under *full admissibility*, estimation function satisfy $\mu_{f,S} \in \varphi_f(S)$ for all f such that $S \in H_f$ and $\mu_{w,a} \in \varphi_w(a)$ for all w such that $a \in F \cup w$. Hence, worse possible matchings are admissible outcomes for each agent, i.e. agents are pessimistic enough. Under this condition, we can establish the next result.

Proposition 2 *Let* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ *be any matching problem with externalities. Assume that agents are pessimistic, i.e.* $\mu_{f,S} \in \varphi_f(S)$ *for all* f *such that* $S \in H_f$ *and* $\mu_{w,a} \in \varphi_w(a)$ *for all* w *such that* $a \in F \cup w$. *Then any stable matching in the reduced problem* $\mu \in \mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^\varphi)$, *is not blocked by any coalition or individual worker in the problem* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

Proposition 2 does not imply that stable matching of the *reduced problem* will be φ-admissible, only ensures that given any of that matchings no agent has incentives to deviate. In the next section, we introduce a constrain on agents' preferences called bottom substitutability that guarantees the existence of φ-stable matchings under *full admissibility.*

3.2 *Bottom substitutability* and the existence of φ-stable matchings.

It is well known that substitutability guarantees the existence of stable matchings in the standard many to one matching problem. However, we argue that externalities and substitutability are in general incompatible concepts. Proposition 1 (and Corollary 1) implies that, we can guarantee the existence of φ-stable matchings by imposing conditions on the problem that ensure the existence of stable matchings in the *reduced problem.* Given a problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, we impose a constrain on preferences called *bottom substitutability,* this condition guarantees that preferences in the corresponding *reduced problem* are substitutable, which implies the existence of at least one stable matching in the *reduced problem.* We require some additional notation.

For each $f \in \mathbf{F}$, let $M_f = \left\{\mu \in \mathcal{M} : \mu = \mu_{f,S} \text{ and } S \in H_f\right\}$ be the set of less preferred matchings for f , given any problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

Definition 5 *For each* $f \in \mathbf{F}$, *and any* $S \subset \mathbf{W}$, *the function* $\Upsilon_f : 2^W \to \mathcal{M}$ *is defined as follows. For any* $S \subset \mathbf{W}$,

1. $\Upsilon_f(S) \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$; *and*
2. $\Upsilon_f(S)\, R_f^* \mu$ *for all* $\mu \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$.

Υ_f is a mapping that selects for any $S \subset \mathbf{W}$, the most preferred matching such that $\mu(f) \subset S$ among the ones in M_f. For instance, if $S = \mathbf{W}$ clearly $\{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f = M_f$, hence $\Upsilon_f(\mathbf{W})$ maps the most preferred matching in M_f. Note that the mapping Υ_f is well defined, since preferences P_f^* are strict and complete for each firm. Even when $S = \phi$, we have that $\Upsilon_f(\phi) = \mu_{f,\phi}$ for each firm.

Suppose that agents are pessimistic in the sense previously described. Note that when we defined agents' preferences to construct an associated the *reduced problem,* we only take into account the order of the less preferred matchings for each agent, for instance given any firm f the preference order P_f^φ over 2^W is completely characterized by the order of the subset of matchings M_f, since for any $S \in H_f$, by construction, there exists a unique matching in the set M_f that satisfies $\mu(f) = S$. The next definition mimics the condition of substitutability defined in the standard matching problem.

Definition 6 *Given a problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, we say that $\mathbf{P}^*$ satisfies bottom substitutability if for each $f \in \mathbf{F}$ and any $S \subset \mathbf{W}$, whenever $w, w' \in S$ and $w \in \mu(f)$ where $\Upsilon_f(S) = \mu$ implies $w \in \mu'(f)$ where $\Upsilon_f(S \setminus \{w'\}) = \mu'$.*

Example 6 *Consider the matching problem analyzed in Example 3.*

$M_{f_1} = \{\mu_5, \mu_6, \mu_7, \mu_9, \mu_{15}, \mu_{18}, \mu_{19}\}$ *and* $M_{f_2} = \{\mu_1, \mu_{11}, \mu_{17}, \mu_{19}\}$.

- *Take $S = \{w_1, w_2, w_3\}$, clearly $\{\mu \in \mathcal{M} : \mu(f_1) \subset S\} \cap M_{f_1} = M_{f_1}$. Since $\mu_5 R_{f_1}^* \mu'$ for all $\mu' \in M_{f_1}$, then $\Upsilon_{f_1}(\{w_1, w_2, w_3\}) = \mu_5$ and $\mu_5(f_1) = \{w_1, w_2\}$.*

- *Take $S \setminus \{w_2\} = \{w_1, w_3\}$, $\{\mu' \in \mathcal{M} : \mu'(f_1) \subset S \setminus \{w_2\}\} \cap M_{f_1} = \{\mu_6, \mu_9, \mu_{19}\}$. Given that $\mu_{19} R_{f_1}^* \mu''$ for all $\mu'' \in \{\mu_6, \mu_9, \mu_{19}\}$, we get $\Upsilon_{f_1}(\{w_1, w_3\}) = \mu_{19}$ and $\mu_{19}(f_1) = \phi$.*

Since $w_1 \notin \mu_{19}(f_1)$, we conclude that bottom substitutability is not satisfied.

Let $\mathcal{BS}$ denote the set of all preference profiles that satisfy *bottom substitutability.*

Theorem 1 *Let $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ be any matching problem with externalities. Suppose that $\mathbf{P}^* \in \mathcal{BS}$, then under full admissibility the set of φ-stable matchings, $\mathbb{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, is not empty.*

We show that the matching problem analyzed in Example 3 has no φ-stable matching and does not satisfied *bottom substitutability*. In the next example, we show a problem that satisfies this condition, we solve the problem with some detail in order to clarify necessary concepts.

Example 7 *Consider a problem of two firms and three workers,* $\mathbf{F} = \{f_1, f_2\}$, $\mathbf{W} = \{w_1, w_2, w_3\}$ *and* $q_{f_1} = q_{f_2} = 2$. *The set of matchings of this problem is given in the Example 2. Preferences are given by the next orders.*

$P^*_{f_1} = \mu_6, \mu_4, \mu_5, \mu_8, \mu_1, \mu_7, \mu_2, \mu_{20}, \mu_{24}, \mu_{23}, \mu_{15}, \mu_{18}, \mu_{22}, \mu_{16}, \mu_{11},$
$\mu_{25}, \mu_{21}, \mu_{19}, \mu_{17}, \mu_{14}, \mu_{13}, \mu_{12}, \mu_3, \mu_{10}, \mu_9.$

$P^*_{f_2} = \mu_{14}, \mu_{13}, \mu_{21}, \mu_{16}, \mu_{12}, \mu_9, \mu_{15}, \mu_2, \mu_{17}, \mu_{19}, \mu_{22}, \mu_{23}, \mu_7, \mu_{10}, \mu_1,$
$\mu_8, \mu_3, \mu_{25}, \mu_{24}, \mu_{20}, \mu_{18}, \mu_6, \mu_5, \mu_4, \mu_{11}.$

$P^*_{w_1} = \mu_1, \mu_2, \mu_3, \mu_5, \mu_6, \mu_7, \mu_8, \mu_4, \mu_9, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{15}, \mu_{16},$
$\mu_{10}, \mu_{17}, \mu_{18}, \mu_{19}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{24}, \mu_{25}, \mu_{20}.$

$P^*_{w_2} = \mu_2, \mu_8, \mu_9, \mu_{12}, \mu_{19}, \mu_{21}, \mu_{23}, \mu_3, \mu_1, \mu_4, \mu_{11}, \mu_{16}, \mu_{18}, \mu_{20}, \mu_{22},$
$\mu_{10}, \mu_5, \mu_6, \mu_7, \mu_{13}, \mu_{15}, \mu_{17}, \mu_{24}, \mu_{25}, \mu_{14}.$

$P^*_{w_3} = \mu_2, \mu_7, \mu_{11}, \mu_{14}, \mu_{17}, \mu_{21}, \mu_{22}, \mu_1, \mu_3, \mu_6, \mu_9, \mu_{15}, \mu_{20}, \mu_{23}, \mu_{24},$
$\mu_{10}, \mu_4, \mu_5, \mu_8, \mu_{13}, \mu_{16}, \mu_{18}, \mu_{19}, \mu_{25}, \mu_{12}.$

In order to check if bottom substitutability is satisfied, it is enough to work with restricted preferences order over sets M_{f_1} *and* M_{f_2}.

$\widetilde{P}^*_{f_1} = \mu_1, \mu_2, \mu_{11}, \mu_{12}, \mu_3, \mu_{10}, \mu_9.$

$\widetilde{P}^*_{f_2} = \mu_9, \mu_2, \mu_{10}, \mu_1, \mu_3, \mu_4, \mu_{11}.$

It is possible to show that these preferences satisfy bottom substitutability:

For instance, if $S = \{w_1, w_2, w_3\}$*:*

$\Upsilon_{f_1}(S) = \mu_1$ *with* $\mu_1(f_1) = \{w_1, w_2\}$*;*

$\Upsilon_{f_1}(S \setminus \{w_1\}) = \mu_{11}$ *with* $\mu_{11}(f_1) = \{w_2\}$*;*

$\Upsilon_{f_1}(S \setminus \{w_2\}) = \mu_2$ *with* $\mu_2(f_1) = \{w_1\}$ *and*

$\Upsilon_{f_1}(S \setminus \{w_3\}) = \mu_1$ *with* $\mu_1(f_1) = \{w_1, w_2\}$.

The preference profile of the reduced problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$ *is,*

$P^{\varphi}_{f_1} = \{w_1, w_2\}, \{w_1\}, \{w_2\}, \phi, \{w_1, w_3\}, \{w_2, w_3\}, \{w_3\}.$

$P^{\varphi}_{f_2} = \{w_1, w_2\}, \{w_2, w_3\}, \{w_1\}, \{w_3\}, \{w_2\}, \phi, \{w_1, w_3\}.$

$P^{\varphi}_{w_1} = f_1, f_2, w_1.$

$P^{\varphi}_{w_2} = f_2, f_1, w_2.$

$P_{w_3}^{\varphi} = f_2, f_1, w_3.$

These preferences are substitutable, if we apply the Deferred acceptance algorithm with the firms proposing (Gale and Shapley, 1962), we get that μ_2 is a stable matching, and by the Corollary 1, $\mu_2 \in \mathbb{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

4 Existence of φ-stable matchings with pessimistic agents.

Hafalir (2008) provides a method to derive a set of *estimation functions* that depends on agents' preferences. He shows that given this *estimation functions,* there exists at least one φ-stable matching. However, this method is strongly based on the fact that there always exist stable matchings in the standard marriage problem. Since in general, there could be instances of the many to one problem with no stable matching, it is not possible to apply Hafalir's (2008) result in this setup.

In this section, we provide a set of *estimation functions* under the assumption that agents are *pessimistic*. Note that, the nonexistence result of Sasaki and Toda (1996) is based on a problem where only one agent is not *pessimistic enough*.

In general the assumption of *full admissibility* is strong, even when under this assumption agents are *pessimistic*, it is not clear why all matchings have to be admissible for each agent. For instance, assume that a firm f is matched with some $S \in H_f$, if agents are pessimistic, $\mu_{f,S} \in \varphi_f(S)$ by assumption. Consider that there is another matching $\mu \in A(f, S)$ that is blocked by any other coalition, hence it is not reasonable that f considers that μ is admissible. Note that when agents are *pessimistic,* this condition seems reasonable to define admissible matchings.

4.1 Admissibility and existence with *pessimistic* agents.

Given any problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, assume that for each firm f such that $S \in H_f$, $\mu_{f,S} \in \varphi_f(S)$. Similarly $\mu_{w,a} \in \varphi_w(a)$ for each w such that $a \in \mathbf{F} \cup w$, i.e. agents are *pessimistic.*

Consider that a firm f is matched with some $S \in H_f$, this firm estimates which matchings are admissible. Since agents are *pessimistic* $\mu_{f,S} \in \varphi_f(S)$, consider the case of any other matching $\mu_{f,S} \neq \mu \in A(f, S)$, if any. Suppose that there is a coalition

$(f', S') \subset \mathbf{F} \cup \mathbf{W} \diagdown (f, S)$ such that $\mu_{f',S'} P^*_{f'} \mu$ and $\mu_{w',f'} P^*_{w'} \mu$ for all $w' \in S'$, i.e. f knows that given μ there is a coalition of agents that has incentives to block the matching, hence μ cannot be admissible for f, i.e. $\mu \notin \varphi_f(S)$. Formally,

Definition 7 *A matching μ is admissible for f (for w), i.e. $\mu \in \rho_f(\mu(f))$ (i.e. $\mu \in \rho_w(\mu(w))$) if there is no coalition $(f', S') \subset \mathbf{F} \cup \mathbf{W} \diagdown (f, \mu(f))$ ($(f', S') \subset \mathbf{F} \cup \mathbf{W} \diagdown (\mu(f), \mu(\mu(f)))$) and a subset of workers $S'' \in \mathbf{W} \diagdown \mu(f)$ ($S'' \in \mathbf{W} \diagdown \mu(\mu(w))$) such that:*

1. *$\mu_{f',S'} P^*_{f'} \mu$ and $\mu_{w',f'} P^*_{w'} \mu$ for all $w' \in S'$; and*

2. *$\mu_{w',w'} P^*_{w'} \mu$ for all $w' \in S''$.*

Let $\boldsymbol{\rho} = \left\{ \left(\rho_f(\cdot), \rho_w(\cdot) \right) : f \in \mathbf{F},\ w \in \mathbf{W} \text{ and } \mathbf{P}^* \right\}$ denote the set of *pessimistic estimation functions* given a problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$. In the next result, we show that given the set of *pessimistic estimation functions* $\boldsymbol{\rho}$, *bottom substitutability* is sufficient to ensure the existence of at least one ρ-stable matching.

Theorem 2 *Let $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ be any matching problem with externalities. Given the set of pessimistic estimation functions $\boldsymbol{\rho}$, if $\mathbf{P}^* \in \mathcal{BS}$ then the set of ρ-stable matchings, $\mathbb{E}_\rho(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, is not empty.*

In the next example, we show that the set of *pessimistic estimation function* $\boldsymbol{\rho}$, implies that not all matchings in the problem are ρ-admissible, then the assumption of *full admissibility* is not a necessary condition for the general existence of ρ-stable matchings.

Example 8 *Consider the matching problem analyzed in the Example 7:*

The set of less preferred matchings for firm f_1 is $M_{f_1} = \{\mu_1, \mu_2, \mu_{11}, \mu_{12}, \mu_3, \mu_{10}, \mu_9\}$, and for firm f_2, $M_{f_2} = \{\mu_9, \mu_2, \mu_{10}, \mu_1, \mu_3, \mu_4, \mu_{11}\}$.

Take the set of matchings $A(f_1, \{w_1, w_2\}) = \{\mu_1, \mu_4\}$, $\mu_1 \in \rho_{f_1}(\{w_1, w_2\})$ by assumption, consider the matching $\mu_4 = f_1, f_1, w_3$. $\mathbf{F} \cup \mathbf{W} \diagdown (f_1, \mu_4(f_1)) = \{f_2, w_3\}$, consider the coalition (f_2, w_3), hence $\mu_{f_2,\{w_3\}} = \mu_1$ and $\mu_{w_3,\{f_2\}} = \mu_1$. Since,

a) $\mu_1 P^*_{f_2} \mu_4$ and

b) $\mu_1 P^*_{w_3} \mu_4$.

Then $\mu_4 \notin \rho_{f_1}(\{w_1, w_2\})$.

Take the set $A(w_1, f_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7, \mu_8\}$, $\mu_4 \in \rho_{w_1}(f_1)$ *by assumption. Take* $\mu_5 = f_1, w_2, w_3$ *and the coalition* $C = (f_2, \{w_2, w_3\})$, *hence* $\mu_{f_2,\{w_2,w_3\}} = \mu_2$, $\mu_{w_2,\{f_2\}} = \mu_3$ *and* $\mu_{w_2,\{f_2\}} = \mu_1$. *Since,*

a) $\mu_2 P^*_{f_2} \mu_5$,

b) $\mu_3 P^*_{w_2} \mu_5$ *and*

c) $\mu_1 P^*_{w_3} \mu_5$.

Then $\mu_5 \notin \rho_{w_1}(f_1)$.

It is possible to determine that the set of pessimistic estimation function for each agent. For firm f_1 *is:*

$\rho_{f_1}(\{w_1, w_2\}) = \{\mu_1\}$;

$\rho_{f_1}(\{w_1, w_3\}) = \{\mu_3\}$;

$\rho_{f_1}(\{w_2, w_3\}) = \{\mu_{10}\}$;

$\rho_{f_1}(\{w_1\}) = \{\mu_2, \mu_7\}$;

$\rho_{f_1}(\{w_2\}) = \{\mu_{11}, \mu_{16}, \mu_{22}\}$;

$\rho_{f_1}(\{w_3\}) = \{\mu_9, \mu_{15}, \mu_{23}\}$ *and*

$\rho_{f_1}(\phi) = \{\mu_{12}, \mu_{13}, \mu_{14}, \mu_{17}, \mu_{19}, \mu_{21}\}$.

For firm f_2*:*

$\rho_{f_2}(\{w_1, w_2\}) = \{\mu_9, \mu_{12}\}$;

$\rho_{f_2}(\{w_1, w_3\}) = \{\mu_{11}\}$;

$\rho_{f_2}(\{w_2, w_3\}) = \{\mu_2\}$;

$\rho_{f_2}(\{w_1\}) = \{\mu_{10}, \mu_{15}, \mu_{16}\}$;

$\rho_{f_2}(\{w_2\}) = \{\mu_3, \mu_8\}$;

$\rho_{f_2}(\{w_3\}) = \{\mu_1, \mu_7\}$ *and*

$\rho_{f_2}(\phi) = \{\mu_4, \mu_5, \mu_6, \mu_{18}, \mu_{20}, \mu_{24}\}$.

For worker w_1*:*

$\rho_{w_1}(f_1) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_7\}$;

$\rho_{w_1}(f_2) = \{\mu_{10}, \mu_{11}, \mu_{12}, \mu_{15}, \mu_{16}\}$; *and*

$\rho_{w_1}(w_1) = \{\mu_{19}, \mu_{20}, \mu_{21}, \mu_{22}\}$.

For worker w_2*:*

$\rho_{w_2}(f_1) = \{\mu_1, \mu_{10}, \mu_{16}, \mu_{22}\}$;

$\rho_{w_2}(f_2) = \{\mu_2, \mu_3, \mu_8, \mu_{12}\}$ *and*

$\rho_{w_2}(w_2) = \{\mu_7, \mu_{13}, \mu_{14}\}$.

For worker w_3*:*

$\rho_{w_3}(f_1) = \{\mu_3, \mu_9, \mu_{10}, \mu_{15}, \mu_{23}\}$;

$\rho_{w_3}(f_2) = \{\mu_1, \mu_2, \mu_7, \mu_{11}\}$ and

$\rho_{w_3}(w_3) = \{\mu_8, \mu_{12}, \mu_{16}, \mu_{19}\}$.

Note that neither of the matchings:

$\{\mu_4, \mu_5, \mu_6, \mu_8, \mu_9, \mu_{11}, \mu_{12}, \mu_{13}, \mu_{14}, \mu_{15}, \mu_{17}, \mu_{18}, \mu_{20}, \mu_{19}, \mu_{20}, \mu_{21}, \mu_{22}, \mu_{23}, \mu_{24}, \mu_{25}\}$,

is ρ-admissible, since each of them is not admissible for at least one agent. The reduced problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\rho})$ has the stable matching $\mu_2 = f_1, f_2, f_2$, It is easy to verify that $\mu_2 \in \rho_a(\mu_2(a))$ for all $a \in \mathbf{F} \cup \mathbf{W}$, hence $\mu_2 \in \mathbb{E}_{\rho}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

The previous example is sufficient to establish the next result.

Proposition 3 *Let $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ be any matching problem with externalities. If agents are pessimistic enough, full admissibility is neither necessary nor sufficient to ensure the general existence of φ-stable matchings.*

4.2 The core with non-myopic agents.

Given a problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ the core denoted by $\mathbb{C}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, is the set of matchings not blocked by any coalition $A \subset \mathbf{F} \cup \mathbf{W}$. Sasaki and Toda (1996) provide an explicit definition of the core, basically a coalition $A \subset \mathbf{F} \cup \mathbf{W}$ blocks the matching μ, if there exist another matching $\mu' \neq \mu$ such that $\mu'(a) \subset A$ and $\mu' P_a^* \mu$ for each $a \in A$, no matter how the agents $\mathbf{F} \cup \mathbf{W} \diagdown \mathbf{A}$ react. It is possible to show that in general the set of pairwise φ-stable matchings and the core does not coincide in the marriage model. Further, Sasaki and Toda (1996) claim that the core may be empty.

Example 9 *Consider a 3x3 marriage problem, the set of all possible matchings is given in Example 1. Preferences are the next orders:*

$P_{m_1}^* = P_{m_3}^* = \mu_6, \mu_4, \mu_2, \mu_1, \mu_5, \mu_3$.

$P_{w_2}^* = P_{w_3}^* = \mu_5, \mu_6, \mu_4, \mu_2, \mu_1, \mu_3$.

$P_{m_2}^* = \mu_2, \mu_5, \mu_6, \mu_4, \mu_1, \mu_3$.

$P_{w_1}^* = \mu_1, \mu_4, \mu_2, \mu_5, \mu_6, \mu_3$.

Under full admissibility $\mathbb{E}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*) = \{\mu_2, \mu_4, \mu_6\}$. It is not difficult to show that μ_2 is blocked by the coalition-matching $(\{m_1.m_3, w_1, w_3\}, \mu_4)$; μ_4 is blocked by $(\{m_1.m_2, w_2, w_3\}, \mu_6)$. Note that μ_6 is φ-stable, in addition any greater coalition which *contain either m_1 or m_3, cannot block μ_6 because this is the most preferred matching for both men. Hence, the core of this game is given by $\mathbb{C}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*) = \{\mu_6\}$.*

The previous example shows that in general the core is contained in the set of pairwise φ-stable matchings, i.e. $\mathbb{C}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*) \subset \mathbb{E}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$. Note that members of a man-woman pair blocks a matching if and only if they are better off under all admissible matchings after deviating. However, under the definition of the core members of a large coalition may deviate from a matching even if there are admissible matchings in which they could be worse off after deviating. Hence, this notion of the core does not consider that some deviating agents may deviate for a second time with another coalition. We conclude that agents deviate in a myopic way under this notion of the core.

For instance, consider the problem in Example 9. The matching μ_2 is φ-stable under full admissibility, but it is blocked by the coalition-matching: $(\{m_1.m_3, w_1, w_3\}, \mu_4)$. Observe that the matching μ_6 is admissible for w_1 and $\mu_2 P^*_{w_1} \mu_6$, this implies that woman w_1 knows that she could be worse off after deviating from μ_2. In addition, w_1 knows that once μ_4 is enforced by $(\{m_1.m_3, w_1, w_3\}, \mu_4)$, the coalition $\{m_1.m_2, w_2, w_3\}$ will block the matching μ_4 and will enforce μ_6, hence if w_1 is not myopic she do not deviate from μ_2.

We analyze an alternative notion of the core by assuming that the agents are not myopic, under this assumption we show that the core with non-myopic agents coincides with the set of φ-stable matchings, and we provide conditions that guarantee the existence of the core with non-myopic agents.

Definition 8 *A coalition $A \subset \mathbf{F} \cup \mathbf{W}$ blocks the matching μ, if there is another matching $\widehat{\mu} \neq \mu$ such that:*

1. *$\widehat{\mu}(a) \subset A$ for all $a \in A$;*
2. *$\mu' P^*_f \mu$ for all $\mu' \in \varphi_f(\widehat{\mu}(f))$ and all $f \in A$; and*
3. *$\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(\widehat{\mu}(w))$ and all $w \in A$.*

Note that according to the previous definition agents do not deviate in a myopic way, even they are members of large coalitions. The core with non-myopic agents given the set of *estimation functions* $\boldsymbol{\varphi}$, or simply the φ-core, is the set of matchings that are not blocked by any coalition $A \subset \mathbf{F} \cup \mathbf{W}$. Let $\mathbb{C}_{\varphi}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ denote the φ-core. We show that for any instance of the matching problem with externalities and given any set of *estimation functions* $\boldsymbol{\varphi}$, the φ-core and the set of φ-stable matchings always coincide. Note that this result is alike the one in the standard problem where the core and the set of stable matchings coincide when all preferences are strict (Roth and Sotomayor, 1990).

Proposition 4 *Let* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ *be any matching problem with externalities and* $\boldsymbol{\varphi}$ *any set of estimation functions, then* $\mathbb{E}_{\varphi}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*) = \mathbb{C}_{\varphi}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$.

Given the previous result, we show that given the set of *pessimistic estimation functions* $\boldsymbol{\rho}$, the ρ-core is always not empty whenever the preferences are *bottom substitutables.,* as in the previous case this result mimics the one of the standard many to one matching problem.

Theorem 3 *Let* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ *be any matching problem with externalities. Given the set of pessimistic estimation functions* $\boldsymbol{\rho}$*, if* $\mathbf{P}^* \in \mathcal{BS}$ *then the* ρ*-core* $\mathbb{C}_{\rho}(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$*, is not empty.*

5 The one to one matching problem.

Note that any many to one matching problem, in which each firm has a quota equal to one, is equivalent to a one to one matching problem. We denote by $\mathbf{M}$ the set of men and by $\mathbf{W}$ the set of women, all necessary notation is trivially extended from the many to one setup. A direct corollary of Theorem 2 is that any marriage problem with externalities has a nonempty set of ρ-stable matchings given the set of *pessimistic estimation functions* $\boldsymbol{\rho}$.

Corollary 2 *Let* $(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$ *be any marriage problem with externalities, given the set of pessimistic estimation functions* $\boldsymbol{\rho}$*, the set of* ρ*-stable matchings* $\mathbb{E}_{\rho}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$*, is not empty.*

The next corollary is a direct consequence of Proposition 4 and Theorem 3, as in the previous case we simply need to assume that each firm has a quota equal to one.

Corollary 3 *Let* $(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$ *be any marriage problem with externalities. Given the set of pessimistic estimation functions* $\boldsymbol{\rho}$*, the* ρ*-core is not empty and is equal to the set of pairwise* ρ*-stable matching, i.e.* $\mathbb{E}_{\rho}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*) = \mathbb{C}_{\rho}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$.

In order to compare our results to Sasaki and Toda's (1996) one, we analyze some examples. We set $|\mathbf{M}| = |\mathbf{W}| \geq 2$ and we restrict agent's preferences over the set of matchings in which there are no unmatched agents, denoted by $\mathbb{M}$. First, we show that whenever agents are *pessimistic, full admissibility* is not necessary for the existence of pairwise φ-stable matchings.

Example 10 *Consider the next 3x3 marriage problem with the next preferences,*

$P^*_{m_1} = P^*_{w_1} : \mu_2, \mu_3, \mu_1, \mu_4, \mu_5, \mu_6.$
$P^*_{m_2} = P^*_{w_2} : \mu_6, \mu_5, \mu_4, \mu_1, \mu_3, \mu_2.$
$P^*_{m_3} = P^*_{w_3} : \mu_4, \mu_1, \mu_5, \mu_3, \mu_6, \mu_2.$

We assume that the agents are pessimistic and we determine which other matching are admissible according to agents' preferences

Consider the set of matchings $A(m_1, w_1) = \{\mu_1, \mu_2\}$*,* $\mu_1 \in \rho_{m_1}(w_1)$ *and* $\mu_1 \in \rho_{w_1}(m_1)$ *by assumption. Take the coalition* (m_2, w_2)*, hence* $\mu^{w_2}_{m_2} = \mu_1$ *and* $\mu^{m_2}_{w_2} = \mu_1$ *given that:*

$\mu_1 P^*_{m_2} \mu_2.$
$\mu_1 P^*_{w_2} \mu_2.$

Then $\mu_2 \notin \rho_{m_1}(w_1) = \{\mu_1\}$ *and* $\mu_2 \notin \rho_{w_1}(m_1) = \{\mu_1\}$.

Applying to the rest of cases we get:

1. $\rho_{m_1}(w_1) = \rho_{w_1}(m_1) = \{\mu_1\}.$
2. $\rho_{m_1}(w_2) = \rho_{w_2}(m_1) = \{\mu_3, \mu_4\}.$
3. $\rho_{m_1}(w_3) = \rho_{w_3}(m_1) = \{\mu_5, \mu_6\}.$
4. $\rho_{m_2}(w_1) = \rho_{w_1}(m_2) = \{\mu_3, \mu_5\}.$
5. $\rho_{m_2}(w_2) = \rho_{w_2}(m_2) = \{\mu_1\}.$
6. $\rho_{m_2}(w_3) = \rho_{w_3}(m_2) = \{\mu_2\}.$
7. $\rho_{m_3}(w_1) = \rho_{w_1}(m_3) = \{\mu_4, \mu_6\}.$
8. $\rho_{m_3}(w_2) = \rho_{w_2}(m_3) = \{\mu_2\}.$
9. $\rho_{m_3}(w_3) = \rho_{w_3}(m_3) = \{\mu_1, \mu_3\}.$

Neither of the matchings $\{\mu_2, \mu_4, \mu_5, \mu_6\}$ *is* ρ*-admissible. Agent's preferences in the reduced problem* $(\mathbf{M}, \mathbf{W}, \mathbf{P}^{\rho})$ *are the next:*

$P_{m_1}^{\rho} = w_1, w_2, w_3,$
$P_{m_2}^{\rho} = w_2, w_1, w_3,$
$P_{m_3}^{\rho} = w_3, w_1, w_2.$
$P_{w_1}^{\rho} = m_1, m_2, m_3,$
$P_{w_2}^{\rho} = m_2, m_1, m_3,$
$P_{w_3}^{\rho} = m_3, m_1, m_2.$

Note that μ_1 is the unique stable matching in the reduced problem $(\mathbf{M}, \mathbf{W}, \mathbf{P}^{\rho})$ *and μ_1 is ρ-admissible, hence μ_1 is ρ-stable.*

In the previous example, we show that not all matchings are ρ-admissible given the set of *pessimistic estimation functions*, this result contrast with the one of Sasaki and Toda (1996). Hafalir (2008) provides a similar result, his characterization is based on an iterative process. At each step, the procedure determines a set of admissible matchings for each man-woman pair. In the first step $t = 1$, each man-woman pair considers admissible the set of stable matchings of the reduced market, in which that pair is not consider. This set of estimations is called *rational expectations estimations,* let $\sigma^1(m, w)$ denote the *estimation function* of the pair (m, w), at $t = 1$. It is well known that the *rational expectations estimations* does not guarantee the existence of φ-stable matchings (Li, 1993). For $t \geq 2$, $\sigma^t(m, w) = \sigma^{t-1}(m, w) \cup I^{\sigma^{t-1}}(m, w)$ where $I^{\sigma^{t-1}}(m, w)$ is the the set of stable matchings of the *reduced problem* given the set of *estimation functions* $\boldsymbol{\sigma}^{t-1} = \{\sigma^{t-1}(m, w) : m \in \mathbf{M} \text{ and } w \in \mathbf{W}\}$, such that m and w are matched. The process finishes when $\sigma(m, w) = \sigma^t(m, w) = \sigma^{t-1}(m, w)$ for each pair (m, w).

For instance, in Example 10, Hafalir's method predicts that μ_1 is the unique σ-stable matching of the problem like in our framework. However, Hafalir's *estimation functions* and the set of *pessimistic estimation functions* do not necessarily coincide.

Note that for any pair (m, w), it is satisfied that $\sigma^{t-1}(m, w) \subset \sigma^t(m, w)$, even if the initial set of *estimation functions* is not the *rational expectations estimations.* Since the set of all possible matchings is finite, Hafalir's iterative process always converge. Further, for any initial set of *estimation functions* $\boldsymbol{\sigma}^1$, the *reduced problem* at each step of the process is a standard marriage problem, hence there exists at least one stable matching. If this matching is not σ^{t-1}-admissible at $t-1$, the process implies that the matchings will be σ^t-admissible at t, then by construction there has to exist at least one σ-stable matching when the process finishes.

For any many to one problem, $(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ and any set of estimation functions $\boldsymbol{\varphi}$ the *reduced problem* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$, is a standard many to one matching problem, hence the set of stable matching might be empty. Even under full admissibility there are problems with no φ-stable matchings, then the *rational expectations estimations* might not be well defined. These characteristics make not possible to apply Hafalir's iterative process to the many to one setup.

Note that Hafalir's process may fail to include a σ-stable matching at the initial set of estimation functions, Hafalir (2008) argues that agents might not know the *estimation functions* of the others, but knows that any stable matching of the *reduced problem* define at each step of the process has to be admissible. Hence, agents update their estimations by including that matchings. However, it is not obvious why rational agents make systematic errors when it is implicitly assume that agents include all relevant information in their estimations. Agents could ignore the *estimation functions* of the others if there is private information, if this is the case, this has to be explicitly specified.

6 Conclusions.

In this paper, we analyze the many to one matching problem with external effects among groups. We argue that standard results about the existence of stable matchings cannot be applied in this setup for at least two reasons. First of all, agents' preferences are defined over different objects, i.e. matchings instead of agents on the other side of the market. This implies that there is no a unique way to order, for instance, subsets of workers hence it is not possible to impose constrains like substitutability which require a well defined preference order over subsets of workers for each firm. On the other hand, the presence of externalities implies that each agent has to take into account how the others react against deviations, hence different behavioral assumptions lead to different notions of the equilibrium.

We extend the notion of stability proposed by Sasaki and Toda (1996) and Hafalir (2008) based on the concept of *estimation functions*. The set of *estimation functions* of a given problem represents, for each agent, a belief about which matchings occur with positive probability given a conjecture about the reactive behavior of agents. In general, *estimation functions* are not uniquely defined and could be exogenously given.

We show that the set φ-stable matchings may be empty, we provide a restriction on preferences called *bottom substitutability* that ensures the existence of at least one φ-stable matching under *full admissibility*.

If agents are *pessimistic* enough, it is possible to construct a set of *pessimistic estimation functions* under a reasonable notion of admissibility. The set of pessimistic estimation functions depends on agent' preferences and guarantees the existence of φ-stable matchings whenever preferences are *bottom substitutable*. In addition, we show that *full admissibility* is neither necessary nor sufficient for the general existence of φ-stable matchings, when agents are *pessimistic* enough.

We analyze the problem of the existence of the core. In general, the core is contained in the set of φ-stable matchings, we argue that the notion of the core is compatible with myopic agents. Given a set of estimation functions $\boldsymbol{\varphi}$, we define a notion of the core with non-myopic agents called φ-core. We show that the set of φ-stable matchings and the φ-core coincide for any $\boldsymbol{\varphi}$, further given the set of *pessimistic estimation functions,* if agents' preferences are *bottom substitutable* the core with non-myopic agents is always not empty. Finally, all results extent to the marriage problem.

7 Appendix: Proofs.

Proposition 1:

Proof. Assume that $\mu \in \mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$ but μ is blocked by some (f, S) such that $S \in H_f$ in the problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$. $\mu' P^{*}_{f} \mu$ for all $\acute{\mu} \in \varphi_f(S)$ implies $\mu^{S}_{f} P^{*}_{f} \mu$, since $\mu \in \varphi_a(\mu(a))$, we know that $\mu R^{*}_{f} \mu^{\mu(f)}_{f}$, hence $\mu^{S}_{f} P^{*}_{f} \mu^{\mu(f)}_{f}$ which implies $S P^{\varphi}_{f}\ \mu(f)$. Assume that there is some $w \in \mu(f) \cap S$, this implies $\mu(w) = \{f\}$ and by assumption $\mu \in \varphi_w(\mu(w))$. Since μ is blocked by (f, S), we know that $\mu'' P^{*}_{w'} \mu$ for all $\mu'' \in \varphi_{w'}(f)$ and all $w' \in S$, it is impossible that $\mu \in \varphi_w(f)$, hence $\mu(f) \cap S = \phi$. $\mu'' P^{*}_{w} \mu$ for all $\mu'' \in \varphi_w(f)$ implies $\mu^{f}_{w} P^{*}_{w} \mu$, since $\mu \in \varphi_a(\mu(a))$ for all $a \in \mathbf{F} \cup \mathbf{W}$, we know that $\mu R^{*}_{w} \mu^{\mu(w)}_{w}$, hence $\mu^{f}_{w} P^{*}_{w} \mu^{\mu(w)}_{w}$ which implies $f P^{\varphi}_{w}\ \mu(w)$ for all $w \in S$. These conditions imply $\mu(f) \neq S$, a) $S P^{\varphi}_{f}\ \mu(f)$ and b) $f P^{\varphi}_{w}\ \mu(w)$ for all $w \in S$, a contradiction.

Now assume μ is blocked by an individual worker $w \in \mathbf{W}$. In this case, we have that $\mu(w) \neq w$ and $\acute{\mu} P^{*}_{w} \mu$ for all $\acute{\mu} \in \varphi_w(w)$. By a similar argument this implies $\mu^{w}_{w} P^{*}_{w} \mu^{\mu(w)}_{w}$, hence $w P^{\varphi}_{w}\ \mu(w)$, a contradiction.

Since $\mu \in \mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$ is φ-admissible and it is not blocked by any worker or coalition, then μ is φ-stable. ■

Proposition 2:

Proof. Suppose that the matching μ is stable in the *reduced problem* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$, and it is blocked by some coalition (f, S) such that $S \in H_f$ in the problem $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$. Hence, $\mu' P_f^{*} \mu$ for all $\acute{\mu} \in \varphi_f(S)$ implies $\mu_f^S P_f^{*} \mu$ and $\mu'' P_w^{*} \mu$ for all $\mu'' \in \varphi_w(f)$ implies $\mu_w^f P_w^{*} \mu$ for all $w \in S$. By assumption, $\mu_{f,\mu(f)} = \mu_f^{\mu(f)} \in \varphi_f(\mu(f))$ and $\mu_{w,\mu(w)} = \mu_w^{\mu(w)} \in \varphi_w(\mu(w))$ hence, it is satisfied $\mu R_f^{*} \mu_f^{\mu(f)}$ and $\mu R_w^{*} \mu_w^{\mu(w)}$ even if μ is not φ-admissible. Then $\mu_f^S P_f^{*} \mu_f^{\mu(f)}$ and $\mu_w^f P_w^{*} \mu_w^{\mu(w)}$ for all $w \in S$, which imply $\mu(f) \neq S$, a) $S P_f^{\varphi}\ \mu(f)$ and b) $f P_w^{\varphi}\ \mu(w)$ for all $w \in S$, a contradiction.

Now suppose that μ is blocked by an individual worker $w \in \mathbf{W}$. In a similar way, we have that $\mu(w) \neq w$ and $\acute{\mu} P_w^{*} \mu$ for all $\acute{\mu} \in \varphi_w(w)$ which implies $\mu_w^w P_w^{*} \mu_w^{\mu(w)}$, hence $w P_w^{\varphi}\ \mu(w)$, a contradiction. ■

Theorem 1:

Proof. Assume that *full admissibility* holds. For each firm $f \in \mathbf{F}$, we define the f's choice function as $Ch_f(S) = \mu(f)$ such that $\Upsilon_f(S) = \mu$ for any $S \subset \mathbf{W}$, obviously this choice function is well defined. First, we have to show that for each $f \in \mathbf{F}$ and any $S \subset \mathbf{W}$ the choice function Ch_f maps the best subset of workers in S according to preferences P_f^{φ}. We know that for any $S \subset \mathbf{W}$, $\Upsilon_f(S) \in M_f$ and $\Upsilon_f(S) R_f^{*} \acute{\mu}$ for all $\mu' \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$, hence by definition $\mu(f) \subset S$ whenever $\Upsilon_f(S) = \mu$. Assume that there is some $S' \subset S$, such that $S' P_f^{\varphi} \mu(f)$ such that $\Upsilon_f(S) = \mu$, hence $S' P_f^{\varphi} \mu(f)$ implies $\mu_f^{S'} P_f^{*} \mu_f^{\mu(f)}$ and by *full admissibility* $\Upsilon_f(S) = \mu = \mu_f^{\mu(f)}$, since $\mu_f^{S'} = \mu_{f,S'} \in \{\mu \in \mathcal{M} : \mu(f) \subset S\} \cap M_f$, a contradiction. Hence, $\mu(f) R_f^{\varphi} S'$ for all $S' \subset S$ where $\Upsilon_f(S) = \mu$, hence $Ch_f(S) = \mu(f)$ such that $\Upsilon_f(S) = \mu$ for any $S \subset \mathbf{W}$ is a well defined choice function for each firm.

We have to show that the preferences profile $\mathbf{P}^{\varphi}$ satisfies the usual substitutability condition. Suppose that $w, \acute{w} \in S$ and $w \in \mu(f)$ where $\Upsilon_f(S) = \mu$, this implies that $w \in Ch_f(S)$. By *bottom substitutability,* we know that $w \in \mu'(f)$ where $\Upsilon_f(S \setminus \{\acute{w}\}) = \acute{\mu}$, this implies that $w \in Ch_f(S \setminus \{\acute{w}\})$, then the preferences profile $\mathbf{P}^{\varphi}$ satisfies substitutability. Hence, the *reduced problem* $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\varphi})$ has a at least one stable matching. Since any matching is φ-admissible by *full admissibility*, then $\mathbb{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$ is not empty by Proposition 1. ■

Theorem 2:

Proof. Let $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{\boldsymbol{\rho}})$ be the *reduced problem* deduced from $(\mathbf{F}, \mathbf{W}, \mathbf{P}^{*})$. By *bottom substitutability*, given the set of *pessimistic estimation functions* $\boldsymbol{\rho}$, there exists at least one stable matching in the *reduced problem*, say $\mu^* \in \mathcal{E}(\mathbf{F}, \mathbf{W}, \mathbf{P}^\rho)$.

We have to show that μ^* is ρ-admissible. Suppose not and $\mu^* \notin \rho_f(\mu^*(f))$ for some firm $f \in \mathbf{F}$. There are two cases:

Case 1: There exists a coalition $(f', S') \subset \mathbf{F} \cup \mathbf{W} \diagdown (f, \mu^*(f))$ such that $\mu_{f',S'} P^*_{f'} \mu$ and $\mu_{w',f'} P^*_{w'} \mu$ for all $w' \in S'$. Since agents are pessimistic, $\mu_{f',S'} = \mu^{S'}_{f'} \in \rho_{f'}(S')$ and $\mu_{w',f'} = \mu^{f'}_{w'} \in \rho_{w'}(f')$, then $\mu^{S'}_{f'} P^*_{f'} \mu^{\mu^*(f')}_{f'}$ and $\mu^{f'}_{w'} P^*_{w'} \mu^{\mu^*(w')}_{w'}$ for all $w' \in S'$. This implies $S' P^\rho_{f'} \mu^*(f')$ and $f' R^\rho_{w'} \mu^*(w')$ for all $w' \in S'$, a contradiction.

Case 2: There exists a subset of workers $S'' \in \mathbf{W} \diagdown \mu(f)$ such that $\mu_{w',w'} P^*_{w'} \mu$ for all $w' \in S''$. In a similar way as before, we know that $\mu_{w',w'} = \mu^{w'}_{w'} \in \rho_{w'}(w')$. Hence, $\mu^{w'}_{w'} P^*_{w'} \mu^{\mu^*(w')}_{w'}$ implies $w' P^\rho_{w'} \mu^*(w')$ for all $w' \in S''$, a contradiction.

Given that f was any firm, this implies that $\mu^* \in \rho_f(\mu^*(f))$ for all $f \in \mathbf{F}$. A similar argument applies for any worker, then the matching μ^* is ρ-admissible, i.e. $\mu^* \in \rho_a(\mu^*(a))$ for all $a \in \mathbf{F} \cup \mathbf{W}$. Hence, by Proposition 1 μ^* is ρ-stable. This completes the proof.

By a different argument, we show that μ^* is ρ-admissible. Since agents are *pessimistic,* by Proposition 2 μ^* is not blocked by any worker and any coalition, hence μ^* is ρ-stable. ■

Proposition 4:

Proof. Suppose that the matching $\mu \in \mathbb{E}_\varphi(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ but $\mu \notin \mathbb{C}_\varphi(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, then there is another matching $\widehat{\mu} \neq \mu$ and a coalition $A \subset \mathbf{F} \cup \mathbf{W}$ which blocks the matching μ. Take any firm $f \in A$ and the subset of worker $\widehat{\mu}(f) \subset A$, obviously $\widehat{\mu}(w) = \{f\} \subset A$ for all $w \in \widehat{\mu}(f)$. It is satisfied: 1) $\mu' P^*_f \mu$ for all $\mu' \in \varphi_f(\widehat{\mu}(f))$ and 2) $\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(f)$ and all $w \in \widehat{\mu}(f)$. Then the coalition $(f, \widehat{\mu}(f))$ blocks the matching μ. If there is no firm in the coalition A, take any worker $w \in A$, obviously $\widehat{\mu}(w) = \{w\} \subset A$ and it is satisfied that: 1) $\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(w)$, hence any individual worker $w \in A$, blocks the matching μ, a contradiction.

On the other side, suppose that the matching $\mu \in \mathbb{C}_\varphi(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$ but $\mu \notin \mathbb{E}_\varphi(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$, then there is at least a coalition, (f, S), or an individual worker, $w \in \mathbf{W}$, which blocks the matching μ. Set the matching $\widehat{\mu}$, such that $\widehat{\mu}(f) = S$, obviously $\widehat{\mu} \neq \mu$, and $A = (f, S)$. By definition is satisfied: 1) $\mu' P^*_f \mu$ for all $\mu' \in \varphi_f(\widehat{\mu}(f))$ and all $f \in A$ and 2) $\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(\widehat{\mu}(w))$ and all $w \in A$, then $\mu \notin \mathbb{C}_\varphi(\mathbf{F}, \mathbf{W}, \mathbf{P}^*)$. Suppose that and individual worker blocks the matching μ, set $A = \{w\}$ and $\widehat{\mu}(w) = w$, obviously $\widehat{\mu} \neq \mu$ and it is satisfied: 1) $\mu'' P^*_w \mu$ for all $\mu'' \in \varphi_w(\widehat{\mu}(w))$ and all $w \in A$, a contradiction. This completes the proof. ■

Theorem 3:

Proof. The proof comes directly from the Theorem 2 and the Proposition 4. ■

Corollary 3:

Proof. Note that the proof of the Proposition 4 is independent of firms' quotas. Consider the set of men equal to the set of firms, the set of workers equal to the set of women and $q_f = 1$ for each $f \in \mathbf{F}$. Then given any set of estimation functions $\boldsymbol{\varphi}$, we have: $\mathbb{E}_{\varphi}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*) = \mathbb{C}_{\varphi}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$. By the Corollary 1, the set of pairwise ρ-stable matchings $\mathbb{E}_{\rho}(\mathbf{M}, \mathbf{W}, \mathbf{P}^*)$, is not empty. ■

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